Testing Nonperturbative Ansätze for the QCD Field Strength Correlator

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Abstract

A test for the Gaussian and exponential Ansätze for the nonperturbative parts of the coefficient functions, $\mathcal{D}^{\rm nonpert.}$ and $\mathcal{D}_1^{\rm nonpert.}$, which parametrize the gauge-invariant bilocal correlator of the field strength tensors in the stochastic vacuum model of QCD, is proposed. It is based on the evaluation of the heavy-quark condensate within this model by making use of the world-line formalism and equating the obtained result to the one following directly from the QCD Lagrangian. This yields a certain relation between $\mathcal{D}^{\rm nonpert.}(0)$ and $\mathcal{D}_1^{\rm nonpert.}(0)$, which is further compared with an analogous relation between these quantities known from the existing lattice data. Such a comparison leads to the conclusion that at the distances smaller than the correlation length of the vacuum, Gaussian Ansatz is more suitable than the exponential one.

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Nowadays, stochastic vacuum model (SVM) [1, 2, 3] is recognized to be one of the most powerful tools for the investigation of both perturbative and nonperturbative phenomena in QCD. Although the predictive power of this model is very high, its field-theoretical status requires further justifications. In fact, an exact derivation of field correlators, which are up to now used in this model as a dynamical input, from the QCD Lagrangian is extremely desirable. Once being performed, such a calculation would shed a new light on an interpolation between the perturbative and nonperturbative effects in QCD. That was the main reason for various authors to address this problem analytically both in QCD [4] and other models possessing the confining phase [5]. However, despite these efforts, the exact form of the coordinate dependence of the bilocal gauge-invariant correlator of field strength tensors in QCD (which plays the major rôle in SVM) remains unknown. The most reliable result known from evaluations of the bilocal correlator on the lattice [6, 7, 8] (see Refs. [9, 10] for a review of the recent progress in the lattice calculations of field strength correlators) is that the nonperturbative parts of the two coefficient functions parametrizing this correlator, $\mathcal{D}^{\text{nonpert.}}$ and $\mathcal{D}_1^{\text{nonpert.}}$, rapidly decrease at the distance, which is usually referred to as the correlation length of the vacuum, T_g. The latter one is equal to 0.13 fm for the $SU(2)$-case [6] and is about 0.22 fm for the $SU(3)$-case [7].

The two standard nonperturbative Ansätze for $\mathcal{D}^{\text{nonpert.}}$ and $\mathcal{D}_1^{\text{nonpert.}}$, used both in lattice investigations and in phenomenological applications of SVM to the evaluation of various QCD processes, are the Gaussian and the exponential ones. In the present Letter, we propose a nontrivial analytical test of these Ansätze. It is based on the evaluation of the heavy-quark condensate by making use of the world-line formalism [11] (see Ref. [12] for a recent review) and further comparison of the obtained result with the one, which follows directly from the QCD Lagrangian [13]. Such a comparison then leads to some relations between $\mathcal{D}^{\text{nonpert.}}(0)$ and $\mathcal{D}_1^{\text{nonpert.}}(0)$ for both Ansätze. Among those, as we shall eventually see, the one obtained for the Gaussian Ansatz satisfies the existing lattice data [7, 8, 9, 10] concerning such a relation much better than the other one. Then, since the typical sizes of the heavy-quark trajectories in the problem under study are of the order of T_g, this leads to the conclusion that at the distances smaller than T_g, Gaussian Ansatz is more suitable than the exponential one. This result is important for future lattice simulations as well as for the applications of SVM to the high-energy hadron scattering [14].

Our method of derivation of the quark condensate is based on the well known formula

$$\left\langle \bar{\psi}\psi \right\rangle = -\frac{1}{V}\frac{\partial}{\partial m}\left\langle \Gamma\left[A_\mu^a\right]\right\rangle_{A_\mu^a}, \tag{1}$$

where V is the four-volume of observation and m is the quark mass. Next, the average $\langle \ldots \rangle_{A_\mu^a}$ on the R.H.S. of Eq. (1) is implied *w.r.t.* the gluodynamics action in the Euclidean space-time, $\frac{1}{4}\int d^4x \left(F_{\mu\nu}^a\right)^2$, with $a = 1, \ldots, N_c^2 - 1$ and $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ being the Yang-Mills field strength tensor. Finally in Eq. (1), $\left\langle \Gamma\left[A_\mu^a\right]\right\rangle_{A_\mu^a}$ is the averaged one-loop quark effective action (*i.e.* one-loop quark self-energy) defined as [11, 12]

$$\left\langle \Gamma\left[A_\mu^a\right]\right\rangle_{A_\mu^a} = -2\int\limits_0^{+\infty}\frac{dT}{T}\mathrm{e}^{-m^2T}\int\limits_P \mathcal{D}x_\mu \int\limits_A \mathcal{D}\psi_\mu \exp\left[-\int\limits_0^T d\tau\left(\frac{1}{4}\dot{x}_\mu^2 + \frac{1}{2}\psi_\mu\dot{\psi}_\mu\right)\right]\times$$

$$\times\left\{\frac{1}{N_c}\left\langle \mathrm{tr}\,\mathcal{P}\exp\left[-ig\int\limits_0^T d\tau\left(A_\mu\dot{x}_\mu - \psi_\mu\psi_\nu F_{\mu\nu}\right)\right]\right\rangle_{A_\mu^a} - 1\right\}. \tag{2}$$

Here, the subscripts P and A denote the periodicity properties of the respective path-integrals, ψ_μ's are antiperiodic Grassmann functions, and $A_\mu \equiv A_\mu^a T^a$ with T^a's standing for the generators of the $SU(N_c)$-group in the fundamental representation. We have also adopted the standard normalization conditions $\langle \Gamma[0] \rangle_{A_\mu^a} = 0$ and $\langle W(0) \rangle_{A_\mu^a} = 1$, where

$$\langle W(C) \rangle_{A_\mu^a} \equiv \frac{1}{N_c} \left\langle \mathrm{tr}\,\mathcal{P} \exp\left(-ig \int\limits_0^T d\tau A_\mu \dot{x}_\mu \right) \right\rangle_{A_\mu^a} \tag{3}$$

is just the Wilson loop, which is defined at the contour C parametrized by the vector $x_\mu(\tau)$. It turns out that the gauge-field dependence of Eq. (2) can be reduced to that of the Wilson loop (3) only. That is because, as it has been demonstrated in Ref. [15], the spin part of the world-line action can be rewritten by means of the operator of the area derivative of the Wilson loop and becomes separated from the average $\langle \ldots \rangle_{A_\mu^a}$:

$$\frac{1}{N_c} \left\langle \mathrm{tr}\,\mathcal{P} \exp\left[-ig \int\limits_0^T d\tau \left(A_\mu \dot{x}_\mu - \psi_\mu \psi_\nu F_{\mu\nu} \right) \right] \right\rangle_{A_\mu^a} = \exp\left(-2 \int\limits_0^T d\tau \psi_\mu \psi_\nu \frac{\delta}{\delta \sigma_{\mu\nu}(x(\tau))} \right) \langle W(C) \rangle_{A_\mu^a}. \tag{4}$$

Within the SVM, the Wilson loop (3) can further we rewritten by virtue of the non-Abelian Stokes theorem and the cumulant expansion as follows [1, 3]

$$\langle W(C) \rangle_{A_\mu^a} \simeq \frac{1}{N_c} \mathrm{tr}\, \exp\left\{ -\frac{1}{2!} \frac{g^2}{4} \int\limits_{\Sigma[C]} d\sigma_{\mu\nu}(z) \int\limits_{\Sigma[C]} d\sigma_{\lambda\rho}(z') \langle\langle F_{\mu\nu}(z) \Phi(z,z') F_{\lambda\rho}(z') \Phi(z',z) \rangle\rangle_{A_\mu^a} \right\}. \tag{5}$$

Here $\langle\langle \mathcal{O}\mathcal{O}' \rangle\rangle_{A_\mu^a} \equiv \langle \mathcal{O}\mathcal{O}' \rangle_{A_\mu^a} - \langle \mathcal{O} \rangle_{A_\mu^a} \langle \mathcal{O}' \rangle_{A_\mu^a}$, and $\Sigma[C]$ is a certain surface bounded by the contour C and parametrized by the vector $z_\mu(\xi)$ with $\xi = (\xi^1, \xi^2)$ standing for the 2D-coordinate. This surface is usually chosen to be the one of the minimal area for a given contour C. We have also denoted for brevity $z \equiv z(\xi)$, $z' \equiv z(\xi')$ and introduced the notation $\Phi(z,z')$ for the parallel transporter factor taken along the straight line joining the points z' and z: $\Phi(z,z') \equiv \frac{1}{N_c} \mathcal{P} \exp\left(-ig \int\limits_{z'}^{z} A_\mu(u) du_\mu \right)$. Next, in a derivation of Eq. (5), the so-called bilocal approximation has been employed, according to which the irreducible gauge-invariant bilocal field strength correlator (cumulant) $\langle\langle F_{\mu\nu}(z)\Phi(z,z')F_{\lambda\rho}(z')\Phi(z',z) \rangle\rangle_{A_\mu^a}$ dominates over all the cumulants of higher orders (see *e.g.* Ref. [16] for details of a derivation of Eq. (5) and discussion of related approximations). Finally, it is worth commenting that the factor $1/2!$ on the R.H.S. of Eq. (5) is simply due to the cumulant expansion, whereas the factor $1/4$ is due to the (non-Abelian) Stokes theorem with the usual agreement on the summation over *all* the indices (not only over those, among which the first one is smaller than the second, used in Refs. [1, 2, 3, 16]).

Let us further parametrize the bilocal cumulant according to the SVM by the two renormalization-group invariant coefficient functions $\mathcal{D}$ and $\mathcal{D}_1$ as follows

$$\langle\langle F_{\mu\nu}(z)\Phi(z,z')F_{\lambda\rho}(z')\Phi(z',z) \rangle\rangle_{A_\mu^a} = \frac{\hat{1}_{N_c \times N_c}}{N_c} \mathcal{N} \Big\{ \left(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda} \right) \mathcal{D}\left((z-z')^2 \right) +$$

$$+\frac{1}{2}\left[\frac{\partial}{\partial z_\mu}\left((z-z')_\lambda\delta_{\nu\rho}-(z-z')_\rho\delta_{\nu\lambda}\right)+\frac{\partial}{\partial z_\nu}\left((z-z')_\rho\delta_{\mu\lambda}-(z-z')_\lambda\delta_{\mu\rho}\right)\right]\mathcal{D}_1\left((z-z')^2\right)\Bigg\}. \quad (6)$$

Here, $\hat{1}_{N_c\times N_c}$ is the unity matrix, and we have used the standard normalization condition for the functions $\mathcal{D}$ and $\mathcal{D}_1$ [3] with $\mathcal{N}\equiv\frac{\left\langle\left(F^a_{\mu\nu}(0)\right)^2\right\rangle_{A^a_\mu}}{24(\mathcal{D}(0)+\mathcal{D}_1(0))}$. By virtue of the (usual) Stokes theorem, the contribution to the double surface integral standing in the argument of the exponent on the R.H.S. of Eq. (5), which emerges from the $\mathcal{D}_1$-part of the cumulant (6), can be rewritten as a boundary term. After that, we eventually arrive at the following expression for the Wilson loop within the SVM

$$\langle W(C)\rangle_{A^a_\mu}=$$

$$=\exp\left\{-\frac{g^2}{8N_c}\mathcal{N}\left[2\int\limits_{\Sigma[C]}d\sigma_{\mu\nu}(z)\int\limits_{\Sigma[C]}d\sigma_{\mu\nu}(z')\mathcal{D}\left((z-z')^2\right)+\oint\limits_C dx_\mu\oint\limits_C dx'_\mu\int\limits_{(x-x')^2}^{+\infty}dt\,\mathcal{D}_1(t)\right]\right\}. \quad (7)$$

Next, one can extract the volume factor V from the heavy-quark self-energy (2) by splitting the coordinate $x_\mu(\tau)$ into the center-of-mass and the relative coordinate [12], $x_\mu(\tau)=\bar{x}_\mu+y_\mu(\tau)$, where the center-of-mass is defined as follows $\bar{x}_\mu=\frac{1}{T}\int\limits_0^T d\tau x_\mu(\tau)$. The empty integration over $\bar{x}$ then obviously yields the factor V. After that, owing to the fact that $\delta x_\mu(\tau)=\delta y_\mu(\tau)$, we can also replace $\frac{\delta}{\delta\sigma_{\mu\nu}(x(\tau))}$ in Eq. (4) by $\frac{\delta}{\delta\sigma_{\mu\nu}(y(\tau))}$, which is possible due to the formula [17] $\frac{\delta}{\delta\sigma_{\mu\nu}(x(\tau))}=\int\limits_{-0}^{+0}d\tau'\tau'\frac{\delta^2}{\delta x_\mu\left(\tau+\frac{1}{2}\tau'\right)\delta x_\nu\left(\tau-\frac{1}{2}\tau'\right)}$.

All the above mentioned considerations lead to the following intermediate expression for the heavy-quark self-energy (2)

$$\left\langle\Gamma\left[A^a_\mu\right]\right\rangle_{A^a_\mu}=-2V\int\limits_0^{+\infty}\frac{dT}{T}\mathrm{e}^{-m^2T}\int\limits_P\mathcal{D}y_\mu\int\limits_A\mathcal{D}\psi_\mu\exp\left[-\int\limits_0^T d\tau\left(\frac{1}{4}\dot{y}_\mu^2+\frac{1}{2}\psi_\mu\dot{\psi}_\mu\right)\right]\times$$

$$\times\left\{\exp\left(-2\int\limits_0^T d\tau\psi_\mu\psi_\nu\frac{\delta}{\delta\sigma_{\mu\nu}(y(\tau))}\right)\times\right.$$

$$\left.\times\exp\left\{-\frac{g^2}{8N_c}\mathcal{N}\left[2\int\limits_{\Sigma[\Gamma]}d\sigma_{\mu\nu}(w)\int\limits_{\Sigma[\Gamma]}d\sigma_{\mu\nu}(w')\mathcal{D}\left((w-w')^2\right)+\oint\limits_\Gamma dy_\mu\oint\limits_\Gamma dy'_\mu\int\limits_{(y-y')^2}^{+\infty}dt\,\mathcal{D}_1(t)\right]\right\}-1\right\}. \quad (8)$$

Here, the contour Γ is parametrized by the vector $y_\mu(\tau)$, and the surface $\Sigma[\Gamma]$, parametrized by the vector $w_\mu(\xi)$, is bounded by this contour.

Next, it can straightforwardly be shown that in the heavy-quark limit under study the typical quark trajectories are small. Indeed, let us consider the free part of the bosonic sector of the world-line action standing in the exponent on the R.H.S. of Eq. (8),

$$\mathcal{S}_{\rm free}=\frac{1}{4}\int\limits_0^T d\tau\dot y_\mu^2(\tau)+m^2T=\frac{1}{2}\int\limits_0^T dt\dot y_\mu^2(t)+\frac{m^2T}{2}.$$

Here, in the last equality we have performed a rescaling $\tau=\frac{t}{2}$, $T^{\rm old}=\frac{T^{\rm new}}{2}$. Among all the reparametrization transformations $t\to\sigma(t)$, $\frac{d\sigma}{dt}\ge 0$, it is convenient to choose the proper-time parametrization $t=\frac{s}{m}$. Here, s is the proper length of the contour Γ so that $T=\frac{L}{m}$, where $L\equiv L[\Gamma]=\int\limits_\Gamma ds\equiv\int\limits_{\sigma_{\rm in}}^{\sigma_{\rm fin}}d\sigma\sqrt{\dot y_\mu^2(\sigma)}$ is just the length of Γ. Within this parametrization, $\int\limits_0^T dt\dot y_\mu^2(t)=m\int\limits_\Gamma ds\left(\frac{dy_\mu(s)}{ds}\right)^2=mL$, since $\left(\frac{dy_\mu(s)}{ds}\right)^2=1$ by the definition of the proper time. Therefore $\mathcal{S}_{\rm free}=mL$, which means that the typical quark trajectories are so that $L\le\frac{1}{m}$, *i.e.* they are really small in the heavy-quark limit.

However, according to the general concepts of the SVM, the distances smaller than the correlation length of the vacuum, T_g, are forbidden for a test quark, *i.e.* L cannot be arbitrarily small. That is because SVM is an effective low-energy theory of QCD with T_g^{-1} playing the rôle of the UV momentum cutoff. Therefore, infinitely small contours C (whose contribution to the Wilson loop (7) would obviously dominate if they are allowed) are forbidden within the SVM. Moreover, C's should be so that not only their lengths obey the inequality $L\ge T_g$, but even among such contours those which lie inside the sphere $S_{T_g/2}(\bar x)$ are forbidden. We conclude that within the SVM, typical heavy-quark trajectories are located from outside to the circle of the radius $T_g/2$ and with the center placed at $\bar x$, nearby to it (in the sense that their lengths L's obey the inequality $L\le\frac{1}{m}$). Therefore, the correlation length of the vacuum is related to the area S inside such a trajectory according to the formula

$$T_g^2\simeq\frac{4}{\pi}S \tag{9}$$

with a good accuracy. We also see that for any two points $y_\mu(\tau)$ and $y_\mu(\tau')$, the following inequality holds

$$|y(\tau)-y(\tau')|\le T_g. \tag{10}$$

Note in particular that since typical sizes of Wilson loops in the problem under study are approximately equal to T_g, our final conclusion on the advantage of the Gaussian Ansatz *w.r.t.* the exponential one will be valid at the distances $|x|\le T_g$.

Let us now evaluate heavy-quark condensate (1) with the Gaussian and exponential Ansätze for the nonperturbative parts $\mathcal{D}^{\rm nonpert.}$ and $\mathcal{D}_1^{\rm nonpert.}$ of the functions $\mathcal{D}$ and $\mathcal{D}_1$. The first of these Ansätze reads $\mathcal{D}\left(x^2\right)=\mathcal{D}(0){\rm e}^{-\frac{x^2}{T_g^2}}$, $\mathcal{D}_1\left(x^2\right)=\mathcal{D}_1(0){\rm e}^{-\frac{x^2}{T_g^2}}$, where from now on we shall denote for brevity $\mathcal{D}^{\rm nonpert.}$ by $\mathcal{D}$ and $\mathcal{D}_1^{\rm nonpert.}$ by $\mathcal{D}_1$. By making use of the result of Ref. [18] for the string tension (see also Ref. [16] for a detailed derivation), we get due to Eq. (9) the following leading $\mathcal{D}$-function contribution

$$2\int\limits_{\Sigma[\Gamma]}d\sigma_{\mu\nu}(w)\int\limits_{\Sigma[\Gamma]}d\sigma_{\mu\nu}(w')\mathcal{D}\left((w-w')^2\right)\simeq 4T_g^2\mathcal{D}(0)\int d^2t{\rm e}^{-t^2}\cdot S\simeq 16\mathcal{D}(0)S^2.$$

For further purposes let us rewrite this result as $8\mathcal{D}(0)\Sigma_{\mu\nu}^2$, where $\Sigma_{\mu\nu}\equiv\Sigma_{\mu\nu}[\Gamma]=\Sigma_{\mu\nu}[C]\equiv\oint\limits_\Gamma y_\mu dy_\nu$ is the tensor area associated with the contour Γ (or C). Here, we have used the fact that

for contours under study, which slightly deviate from the circle, $S^2 = \frac{1}{2}\Sigma_{\mu\nu}^2$. By virtue of Eq. (10), we also have

$$\oint_\Gamma dy_\mu \oint_\Gamma dy'_\mu \int_{(y-y')^2}^{+\infty} dt\, \mathcal{D}_1(t) = \mathcal{D}_1(0) T_g^2 \oint_\Gamma dy_\mu \oint_\Gamma dy'_\mu \mathrm{e}^{-\frac{(y-y')^2}{T_g^2}} \simeq$$

$$\simeq \mathcal{D}_1(0) T_g^2 \cdot \frac{2}{T_g^2} \oint_\Gamma dy_\mu \oint_\Gamma dy'_\mu y_\nu y'_\nu = 2\mathcal{D}_1(0)\Sigma_{\mu\nu}^2.$$

This leads to the following expression for the Wilson loop standing on the R.H.S. of Eq. (8)

$$\exp\left\{-\frac{g^2}{8N_c}\mathcal{N}\left[2\int_{\Sigma[\Gamma]} d\sigma_{\mu\nu}(w) \int_{\Sigma[\Gamma]} d\sigma_{\mu\nu}(w')\mathcal{D}\left((w-w')^2\right) + \oint_\Gamma dy_\mu \oint_\Gamma dy'_\mu \int_{(y-y')^2}^{+\infty} dt\, \mathcal{D}_1(t)\right]\right\} \simeq$$

$$\simeq \mathrm{e}^{-\mathcal{A}_{\mathrm{Gauss}}\Sigma_{\mu\nu}^2}, \quad \text{where} \quad \mathcal{A}_{\mathrm{Gauss}} \equiv \frac{\pi^2}{24N_c}\frac{4\mathcal{D}(0)+\mathcal{D}_1(0)}{\mathcal{D}(0)+\mathcal{D}_1(0)} \cdot \frac{\alpha_s}{\pi}\left\langle \left(F_{\mu\nu}^a(0)\right)^2\right\rangle_{A_\mu^a} \qquad (11)$$

with $\alpha_s \equiv \frac{g^2}{4\pi}$. Note that the behaviour of small Wilson loops of the type $\mathrm{e}^{-\mathrm{const.}S^2}$ was for the first time obtained in Ref. [2].

In the same way one can treat the exponential Ansatz, $\mathcal{D}(x^2) = \mathcal{D}(0)\mathrm{e}^{-\frac{\sqrt{x^2}}{T_g}}$, $\mathcal{D}_1(x^2) = \mathcal{D}_1(0)\mathrm{e}^{-\frac{\sqrt{x^2}}{T_g}}$. In that case we have: $2\int_{\Sigma[\Gamma]} d\sigma_{\mu\nu}(w) \int_{\Sigma[\Gamma]} d\sigma_{\mu\nu}(w')\mathcal{D}\left((w-w')^2\right) \simeq 16\mathcal{D}(0)\Sigma_{\mu\nu}^2$,

$$\oint_\Gamma dy_\mu \oint_\Gamma dy'_\mu \int_{(y-y')^2}^{+\infty} dt\, \mathcal{D}_1(t) = \mathcal{D}_1(0)\oint_\Gamma dy_\mu \oint_\Gamma dy'_\mu \int_{(y-y')^2}^{+\infty} dt\, \mathrm{e}^{-\frac{\sqrt{t}}{T_g}} = 2\mathcal{D}_1(0)T_g^2 \times$$

$$\times \oint_\Gamma dy_\mu \oint_\Gamma dy'_\mu \left(1 + \frac{|y-y'|}{T_g}\right)\mathrm{e}^{-\frac{|y-y'|}{T_g}} \simeq 2\mathcal{D}_1(0)T_g^2 \oint_\Gamma dy_\mu \oint_\Gamma dy'_\mu \left(1 - \frac{(y-y')^2}{T_g^2}\right) = 4\mathcal{D}_1(0)\Sigma_{\mu\nu}^2.$$

Here in the evaluation of the $\mathcal{D}$- and $\mathcal{D}_1$-dependent parts, Eqs. (9) and (10), respectively, have been applied. We see that for the exponential Ansatz, Eq. (11) remains valid with the replacement $\mathcal{A}_{\mathrm{Gauss}} \to \mathcal{A}_{\mathrm{exp}} = 2\mathcal{A}_{\mathrm{Gauss}}$.

In order to proceed with the evaluation of the path-integral (8), it is useful to linearize the quadratic $\Sigma_{\mu\nu}$-dependence of Eq. (11) by introducing the integration over an auxiliary antisymmetric tensor field. Clearly, since $\Sigma_{\mu\nu}$ depends on the contour Γ as a whole, this field will be space-time independent. Introducing for brevity the common notation $\mathcal{A}$ for both $\mathcal{A}_{\mathrm{Gauss}}$ and $\mathcal{A}_{\mathrm{exp}}$, we have

$$\exp\left(-\mathcal{A}\sum_{\mu,\nu=1}^{4}\Sigma_{\mu\nu}^2\right) = \exp\left(-2\mathcal{A}\sum_{\mu<\nu}\Sigma_{\mu\nu}^2\right) = \frac{1}{(8\pi\mathcal{A})^3}\int_{-\infty}^{+\infty}\prod_{\mu<\nu} dB_{\mu\nu}\exp\left(-\frac{B_{\mu\nu}^2}{8\mathcal{A}} - iB_{\mu\nu}\Sigma_{\mu\nu}\right) =$$

$$= \frac{1}{(8\pi\mathcal{A})^3}\int_{-\infty}^{+\infty}\left(\prod_{\mu<\nu} dB_{\mu\nu}\mathrm{e}^{-\frac{B_{\mu\nu}^2}{8\mathcal{A}}}\right)\exp\left(-\frac{i}{2}\sum_{\mu,\nu=1}^{4} B_{\mu\nu}\Sigma_{\mu\nu}\right).$$

Note that only in this equation we have emphasized explicitly the summation over indices in order to avoid possible misleadings. In what follows, in the expressions of the type $\mathcal{O}_{\mu\nu}\mathcal{O}'_{\mu\nu}$ we shall as usual assume the summation over all the values of indices.

The one-loop heavy-quark self-energy (8) has now reduced to that in the constant field $B_{\mu\nu}$, which should eventually be averaged over. Indeed, since due to the Stokes theorem, $B_{\mu\nu}\Sigma_{\mu\nu} = \int\limits_{\Sigma[\Gamma]} d\sigma_{\mu\nu}(w)B_{\mu\nu}(w)$, the following equality holds (*cf.* Ref. [15]) $\exp\left(-2\int\limits_0^T d\tau\psi_\mu\psi_\nu\frac{\delta}{\delta\sigma_{\mu\nu}(y(\tau))}\right)\mathrm{e}^{-\frac{i}{2}B_{\mu\nu}\Sigma_{\mu\nu}} = \exp\left(i\int\limits_0^T d\tau B_{\mu\nu}\psi_\mu\psi_\nu\right)$, we get upon the insertion of Eq. (11) into Eq. (8) the following expression

$$\left\langle\Gamma\left[A_\mu^a\right]\right\rangle_{A_\mu^a} = -\frac{2V}{(8\pi\mathcal{A})^3}\int\limits_{-\infty}^{+\infty}\left(\prod_{\mu<\nu}dB_{\mu\nu}\mathrm{e}^{-\frac{B_{\mu\nu}^2}{8\mathcal{A}}}\right)\int\limits_0^{+\infty}\frac{dT}{T}\mathrm{e}^{-m^2T}\times$$

$$\times\left\{\int\limits_P\mathcal{D}y_\mu\int\limits_A\mathcal{D}\psi_\mu\exp\left[-\int\limits_0^T d\tau\left(\frac{1}{4}\dot{y}_\mu^2+\frac{1}{2}\psi_\mu\dot{\psi}_\mu-\frac{i}{2}B_{\mu\nu}y_\mu\dot{y}_\nu-iB_{\mu\nu}\psi_\mu\psi_\nu\right)\right]-\frac{1}{(4\pi T)^{\frac{D}{2}}}\right\}. \quad (12)$$

Taking now into account the known one-loop expression for the Euler-Heisenberg Lagrangian in spinor QED, *i.e.* one-loop electron effective action in a constant background field (see *e.g.* [12]), we have

$$\int\limits_P\mathcal{D}y_\mu\int\limits_A\mathcal{D}\psi_\mu\exp\left[-\int\limits_0^T d\tau\left(\frac{1}{4}\dot{y}_\mu^2+\frac{1}{2}\psi_\mu\dot{\psi}_\mu-\frac{i}{2}B_{\mu\nu}y_\mu\dot{y}_\nu-iB_{\mu\nu}\psi_\mu\psi_\nu\right)\right]-\frac{1}{(4\pi T)^{\frac{D}{2}}} =$$

$$= \frac{1}{(4\pi T)^{\frac{D}{2}}}\left[T^2ab\cot(aT)\coth(bT)-1\right]. \quad (13)$$

Here, the standard notations were adopted: $a^2 = \frac{1}{2}\left[\mathbf{E}^2-\mathbf{H}^2+\sqrt{\left(\mathbf{E}^2-\mathbf{H}^2\right)^2+4(\mathbf{E}\cdot\mathbf{H})^2}\right]$, $b^2 = \frac{1}{2}\left[-\left(\mathbf{E}^2-\mathbf{H}^2\right)+\sqrt{\left(\mathbf{E}^2-\mathbf{H}^2\right)^2+4(\mathbf{E}\cdot\mathbf{H})^2}\right]$ with $\mathbf{E}\equiv i\left(B_{14},B_{24},B_{34}\right)$, $\mathbf{H}\equiv\left(B_{23},-B_{13},B_{12}\right)$.

Next, due to the factor e^{-m^2T} in Eq. (12), in the heavy-quark limit under study only small values of the proper time T are sufficient, and the R.H.S. of Eq. (13) can be expanded in powers of T. Then, the leading term of this expansion, which in the expansion of the fermionic determinant corresponds to the diagram with two external legs of the $B_{\mu\nu}$-field, reads $T^2ab\cot(aT)\coth(bT)-1 = \frac{T^2}{3}\left(b^2-a^2\right)+\mathcal{O}\left(T^4(\mathbf{E}\cdot\mathbf{H})^2\right)$. This leads to the following expression for the heavy-quark self-energy (12)

$$\left\langle\Gamma\left[A_\mu^a\right]\right\rangle_{A_\mu^a} = -\frac{2V}{(8\pi\mathcal{A})^3}\int\limits_{-\infty}^{+\infty}\left(\prod_{\mu<\nu}dB_{\mu\nu}B_{\mu\nu}^2\mathrm{e}^{-\frac{B_{\mu\nu}^2}{8\mathcal{A}}}\right)\cdot\frac{1}{3}\int\limits_0^{+\infty}dTT\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^{\frac{D}{2}}}.$$

Next, the pole at $D = 4$ emerging in the integral over proper time, $\int\limits_0^{+\infty}dTT\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^{\frac{D}{2}}} = \frac{m^{D-4}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)$, can be subtracted by making use of the MS-prescription (Clearly, the additional constant, which appears if we use *e.g.* the $\overline{MS}$-prescription, does not contribute to the

quark condensate, since the derivative of this constant *w.r.t.* m in Eq. (1) vanishes.). This finally yields

$$\left\langle \Gamma\left[A_\mu^a\right]\right\rangle_{A_\mu^a} = -\frac{2V}{(8\pi\mathcal{A})^3}\frac{1}{3(4\pi)^2}\ln\frac{\Lambda^2}{m^2}\cdot\int\limits_{-\infty}^{+\infty}\prod_{\mu<\nu} dB_{\mu\nu}B_{\mu\nu}^2 \mathrm{e}^{-\frac{B_{\mu\nu}^2}{8\mathcal{A}}} =$$

$$= -\frac{2V}{(8\pi\mathcal{A})^3}\frac{1}{3(4\pi)^2}\ln\frac{\Lambda^2}{m^2}\cdot\left[2^{\frac{7}{2}}\pi^{\frac{1}{2}}\mathcal{A}^{\frac{3}{2}}\cdot(8\pi\mathcal{A})^{\frac{5}{2}}\cdot 6\right] = -\frac{2V\mathcal{A}}{\pi^2}\ln\frac{\Lambda}{m}$$

with Λ standing for the UV momentum cutoff. Owing to Eq. (1), at $N_c = 3$ we arrive at the following value of the heavy-quark condensate $\left\langle\bar\psi\psi\right\rangle = -\frac{2\mathcal{A}}{\pi^2 m}$, which should be equal to the result following directly from the QCD Lagrangian (*i.e.* from the corresponding triangle diagram) [13] $\left\langle\bar\psi\psi\right\rangle = -\frac{1}{12m}\cdot\frac{\alpha_s}{\pi}\left\langle\left(F_{\mu\nu}^a(0)\right)^2\right\rangle_{A_\mu^a}$. For $\mathcal{A} = \mathcal{A}_{\mathrm{Gauss}}$, this yields the following relation between the nonperturbative parts of the functions $\mathcal{D}$ and $\mathcal{D}_1$ at the origin: $\mathcal{D}_1(0) = \frac{1}{2}\mathcal{D}(0)$. Similarly, for $\mathcal{A} = \mathcal{A}_{\mathrm{exp}}$, we obtain $\mathcal{D}_1(0) = 5\mathcal{D}(0)$. These relations should be compared with the results of those lattice measurements [7, 8, 9, 10], whose χ^2 is minimal. Such results can be summarized by the table, following from the data reviewed in Ref. [10], which has the form

Theory	$\mathcal{D}_1(0)/\mathcal{D}(0)$
Quenched approximation at the distances $0.1\,\mathrm{fm} \le \lvert x\rvert \le 1\,\mathrm{fm}$	0.22 ± 0.05
Quenched approximation at the distances $0.4\,\mathrm{fm} \le \lvert x\rvert \le 1\,\mathrm{fm}$	0.29 ± 0.13
The same as in the previous line, but with the perturbative-like part of the fit fixed to zero	0.34 ± 0.04
Full QCD with the quark mass (in lattice units) equal to 0.01	0.13 ± 0.08
The same as in the previous line, but with the quark mass equal to 0.02	0.13 ± 0.07
Average over the above values	0.20 ± 0.10

This table indicates that the result obtained within the Gaussian Ansatz is closer to the existing lattice data than the other one.

In conclusion of the present Letter, we have proposed a nontrivial analytical test for the Gaussian and exponential Ansätze for the nonperturbative parts of two coefficient functions, which parametrize the gauge-invariant bilocal field strength correlator in QCD within the SVM. It was based on the evaluation of a heavy-quark condensate within this model by making use of the world-line formalism and further comparison of the obtained result with the exact one, following directly from the QCD Lagrangian. The outcome of this investigation suggests that the Gaussian Ansatz satisfies the existing lattice data better than the exponential one. This means that at the distances $\lvert x\rvert \le T_g$, the former Ansatz is more favourable than the latter one. Such a conclusion is important for further applications of the SVM to the evaluation of various QCD processes as well as for the lattice measurements of field correlators.

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